EFFECTIVE ELASTIC MODULI OF A FIBROUS COMPOSITE MATERIAL, ISOTROPIC ON THE AVERAGE

V. V. Kolokol'chikov and G. N. Sapego UDC 678:539:376

There exist a number of methods for finding the effective moduli of composite materials (see, for example, [1, 2]). The exact calculation of the effective moduli of a composite material is difficult. The anisotropic effective elastic moduli of several fibrous structures with transversally isotropic or orthotropic symmetry were found in a correlation approximation in [3, 4], using the Royce scheme in [5], as well as in [6] on the basis of a simplified variant of the theory of [7]. There exist [8] simple formulas for the effective Young modulus of a composite material made of fiber, in the cylindrical shell of the matrix, in the case of axial loading and a single-row distribution of the pressure.

§1. Let us consider a composite material, chaotically reinforced with isotropic fibers. As a result of the chaotic character of the orientation of the fibers, the composite material will be isotropic on the average.

The indices c, f, m denote quantities characterizing the composite material, a fiber, and a matrix, respectively. The effective elastic moduli are obtained on the basis of a generalized rule of mixing, with consecutive and parallel addition of some elements. For this purpose, we introduce N_f (N_f is the number of fibers in the composite material) elements, consisting of a fiber, surrounded by an effective cylindrical layer of the material of the matrix. The volume of the effective layer is equal to the volume of the matrix V_m^a divided by N_f.

The expression for the effective density of the free energy, neglecting the interaction between a resin and a boric fiber, is the sum of the energies of the matrix and the isotropic fiber:

$$F = \sum_{\xi=m,f} \frac{V_\xi}{3}\left[\frac{\lambda_\xi}{2}\sum_{j=1}^{3}\left(\varepsilon_{ii}^{\prime(\xi j)}\right)^2 + \mu_\xi \sum_{k=1}^{3} \varepsilon_{ij}^{\prime(\xi k)}\varepsilon_{ij}^{\prime(\xi k)} - (3\lambda_\xi + 2\mu_\xi)\,\alpha_\xi\,\vartheta \sum_{j=1}^{3} \varepsilon_{ii}^{\prime(\xi j)}\right], \quad (1.1)$$

where λ_m, μ_m, λ_f, μ_f are the Lamé constants of the matrix and the fiber, depending on the temperature; V_m, V_f are the relative volumetric contents of the matrix and the fiber in the composite material; α_m, α_f are the coefficients of linear thermal expansion of the matrix and the fiber; $\vartheta = T - T_0$ is the temperature drop; T_0 is the temperature at which there are no stresses or deformations; T is the absolute equilibrium temperature, not depending on the coordinates; $\varepsilon_{ij}^{\prime(mk)}$ (i, j, k = 1, 2, 3) are the components of the deformations of the matrix for elements of the k-th kind, being the mean elements over all the parallel or consecutive elements. In (1.1) the effective density of the free energy of the composite material is found by superposition of the energies of elements depending on the deformations, different for different elements and different components of the composite material. In (1.1) and in what follows the summation is carried out over all the repeating Roman subscripts.

As a result of the postulation of the isotropy of the composite material on the average, the manner of writing (1.1) corresponds to three (k = 1, 2, 3) equally justified kinds of middle elements. The elements of each kind are parallel to each other, and perpendicular to elements of another kind.

We direct the axes of the three types of elements along the axes of coordinates; then if, relative to some arbitrary loading, the elements of one type are parallel one to another, the elements of the two other types, both within the type of elements and between themselves, will be connected in sequence.

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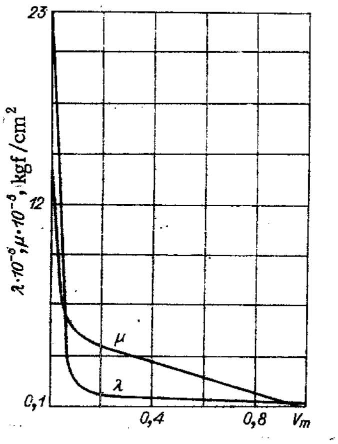

Fig. 1

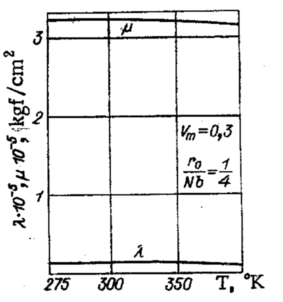

Fig. 2

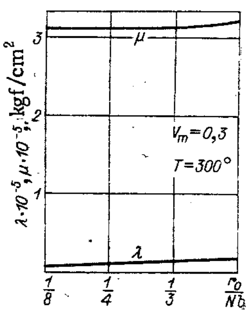

Fig. 3

We denote by $\varepsilon_{ij}^{(mk)}$, $\varepsilon_{ij}^{(fk)}$ (k = 1, 2, 3) the components of the tensors of the deformations (below called elementary deformations), averaged over an element of the k-th type, respectively, for the matrix and the fiber, for which the rules for the summation of the deformations of elements participating in a consecutive connection are valid; to obtain the deformations of the composite material, the rule of the equality of the deformations of elements, participating in a parallel connection, to the deformations of the composite material is also valid.

The deformations of the composite material, to which the elements offer a resistance, being connected in parallel, are equal to the elementary deformations of the matrix and the fiber. The deformations of the composite material, which are resisted by two types of elements, connected consecutively, are equal to the sum of the elementary deformations of the matrices and the fibers of the elements of these two types, i.e.,

$$\varepsilon_{\alpha\alpha}^{(c)} = \frac{N_f}{3}\left(\varepsilon_{\alpha\alpha}^{(m\beta)} + \varepsilon_{\alpha\alpha}^{(f\beta)} + \varepsilon_{\alpha\alpha}^{(m\gamma)} + \varepsilon_{\alpha\alpha}^{(f\gamma)}\right) = \varepsilon_{\alpha\alpha}^{(m\alpha)} = \varepsilon_{\alpha\alpha}^{(f\alpha)},$$
$$\varepsilon_{\alpha\beta}^{(c)} = \frac{N_f}{3}\left(\varepsilon_{\alpha\beta}^{(m\alpha)} + \varepsilon_{\alpha\beta}^{(f\alpha)} + \varepsilon_{\alpha\beta}^{(mf)} + \varepsilon_{\alpha\beta}^{(f\beta)}\right) = \varepsilon_{\alpha\beta}^{(m\gamma)} = \varepsilon_{\alpha\beta}^{(f\gamma)}, \tag{1.2}$$
$$\alpha,\ \beta,\ \gamma = 1,\ 2,\ 3;\ \alpha \neq \beta \neq \gamma.$$

In equalities (1.2) and in what follows summation is not carried out over the repeating Greek subscripts. The mean deformations of elements of one kind are assumed identical for a region of identical components.

To determine the effective elastic moduli it is sufficient to find them during the course of any arbitrary processes. With arbitrary processes, expression (1.1) can correspond to anisotropic materials if the following equalities are not satisfied for them

$$\varepsilon_{ii}^{\prime(\xi\alpha)} = \varepsilon_{ii}^{\prime(\xi\beta)} = \varkappa^{\xi}\varepsilon_{ii}^{(c)}, \quad \xi = m, f; \tag{1.3}$$
$$\varepsilon_{ij}^{\prime(\xi\alpha)}\varepsilon_{ij}^{\prime(\xi\alpha)} = \varepsilon_{ij}^{\prime(\xi\beta)}\varepsilon_{ij}^{\prime(\xi\beta)} = \eta^{\xi}\varepsilon_{ij}^{(c)}\varepsilon_{ij}^{(c)} + \zeta^{\xi}\left(\varepsilon_{ii}^{(c)}\right)^2,$$
$$\alpha,\ \beta = 1,\ 2,\ 3$$
$$\varkappa^{\xi} = \text{const},\ \eta^{\xi} = \text{const},\ \zeta^{\xi} = \text{const}.$$

The possibility that (1.3) will not be satisfied is a consequence of the nonequivalent replacement of an isotropic composite material by a system of three types of elements. To eliminate this nonequivalence, we limit ourselves to processes leading to (1.3). As such processes, let us consider two, for which

$$\varepsilon_{11}^{(c)} = \varepsilon_{22}^{(c)} = \varepsilon_{33}^{(c)} \neq 0; \quad \varepsilon_{12}^{(c)} = \varepsilon_{13}^{(c)} = \varepsilon_{23}^{(c)} = 0; \tag{1.4}$$
$$\varepsilon_{11}^{(c)} = \varepsilon_{22}^{(c)} = \varepsilon_{33}^{(c)} = 0; \quad \varepsilon_{12}^{(c)} = \varepsilon_{13}^{(c)} = \varepsilon_{23}^{(c)} \neq 0. \tag{1.5}$$

For the quantities entering into (1.2), as a result of the equivalance of the axes, we can set

$$\varepsilon_{\alpha\alpha}^{(\xi\beta)} = \varepsilon_{\alpha\alpha}^{(\xi\gamma)}; \quad \varepsilon_{\alpha\beta}^{(\xi\alpha)} = \varepsilon_{\alpha\beta}^{(\xi\beta)}, \quad \xi = m, f, \quad \alpha \neq \beta. \tag{1.6}$$

The deformations $\varepsilon_{ij}^{\prime(\xi k)}$, entering into (1.1), depend on the elementary deformations $\varepsilon_{ij}^{(\xi k)}$ with the processes (1.4), (1.5) in the same way as the deformations of the composite material $\varepsilon_{ij}^{(c)}$ depend on $\varepsilon_{ij}^{(mk)}$ or on $\varepsilon_{ij}^{(fk)}$ in the case of the processes (1.4), (1.5), where the composite material is a monomer of m or f. The latter assertion stems from the fact that, in the case of a monomer, $\varepsilon_{ij}^{\prime(\xi k)}$ coincides with $\varepsilon_{ij}^{(c)}$ and, of the deformations $\varepsilon_{ij}^{(\xi k)}$, we have only $\varepsilon_{ij}^{(mk)}$ or $\varepsilon_{ij}^{(fk)}$.

From this, and from (1.2), (1.6), it follows that the elementary deformations and the deformations of elements of some type entering into (1.1) will be equal to each other if they correspond to loads, relative to which the elements are connected in a parallel manner. If the elements are connected in a consecutive manner, then, from (1.1), deformations corresponding to the loading of consecutive elements will be greater than the elementary deformations by $(2/3)N_f$ times ($(2/3)N_f$ is the number of consecutive elements). Thus, the deformations $\varepsilon'^{(\xi k)}_{ij}$ are defined in terms of the elementary deformations using the relationships

$$\varepsilon'^{(\xi k)}_{\alpha\alpha} = \begin{cases} \varepsilon^{(\xi\alpha)}_{\alpha\alpha} & \text{with } k = \alpha, \\ \frac{2}{3} N_f \varepsilon^{(\xi\beta)}_{\alpha\alpha} & \text{with } k = \beta, \\ \frac{2}{3} N_f \varepsilon^{(\xi\gamma)}_{\alpha\alpha} & \text{with } k = \gamma,\ \xi = m, f; \end{cases} \tag{1.7}$$

$$\varepsilon'^{(\xi k)}_{\alpha\beta} = \begin{cases} \frac{2}{3} N_f \varepsilon^{(\xi\alpha)}_{\alpha\beta} & \text{with } k = \alpha, \\ \frac{2}{3} N_f \varepsilon^{(\xi\beta)}_{\alpha\beta} & \text{with } k = \beta, \\ \varepsilon^{(\xi\gamma)}_{\alpha\beta} & \text{with } k = \gamma,\ \xi = m, f, \end{cases}$$

$$\alpha, \beta, \gamma = 1, 2, 3, \ \alpha \neq \beta \neq \gamma.$$

With the process (1.4), (1.6), satisfaction of relationships (1.3), taking account of (1.7), (1.2) leads to the equalities

$$(1 - \varkappa)\, \varepsilon^{(c)}_{\alpha\alpha} = \frac{2}{3} N_f \varepsilon^{(f\beta)}_{\alpha\alpha}, \quad \alpha \neq \beta, \tag{1.8}$$

$$\varkappa \varepsilon^{(c)}_{\alpha\alpha} = \frac{2}{3} N_f \varepsilon^{(m\beta)}_{\alpha\alpha}, \quad \alpha \neq \beta,$$

$$\varkappa^f = (3 - 2\varkappa)/3, \ \varkappa^m = (1 + 2\varkappa)/3,$$

$$\zeta^\xi = (1/3)[(\varkappa^\xi)^2 - \eta^\xi].$$

In the case of the process (1.5), (1.6), from (1.3), (1.7), (1.2) it follows

$$(1 - \eta)\, \varepsilon^{(c)}_{\alpha\beta} = \frac{2}{3} N_f \varepsilon^{(f\alpha)}_{\alpha\beta}, \quad \alpha \neq \beta, \tag{1.9}$$

$$\eta \varepsilon^{(c)}_{\alpha\beta} = \frac{2}{3} N_f \varepsilon^{(m\alpha)}_{\alpha\beta}, \quad \alpha \neq \beta,$$

$$\eta^f = [1 + 2(1 - \eta)^2]/3, \ \eta^m = [1 + 2\eta^2]/3.$$

The deformations $\varepsilon'^{(\xi\beta)}_{\alpha\alpha}$ and $\varepsilon'^{(\xi\alpha)}_{\alpha\beta}$ ($\alpha \neq \beta$) are connected with the true mean deformation of the ξ-component and of the elements participating in the consecutive connection, and with the volumetric contents of the components by the relationships

$$\varepsilon'^{(\xi\beta)}_{\alpha\alpha} \equiv \frac{2}{3} N_f \varepsilon^{(\xi\beta)}_{\alpha\alpha} = V_\xi \varepsilon^{\xi\beta}_{\alpha\alpha}; \quad \varepsilon'^{(\xi\alpha)}_{\alpha\beta} = \frac{2}{3} N_f \varepsilon^{(\xi\alpha)}_{\alpha\beta} = V_\xi \varepsilon^{\xi\alpha}_{\alpha\beta}, \tag{1.10}$$

$$\xi = f, m, \ V_f + V_m = 1.$$

Formulas (1.10) follow from (1.7), respectively, with $k = \beta, \gamma$ and $k = \alpha, \beta$, since $\varepsilon'^{(\xi\beta)}_{\alpha\alpha}$ and $\varepsilon'^{(\xi\alpha)}_{\alpha\beta}$ are deformations, averaged over the volume of the consecutive elements, and $\varepsilon^{\xi\beta}_{\alpha\alpha}$ and $\varepsilon^{\xi\alpha}_{\alpha\beta}$ are deformations, averaged over the volume of the fibers and the matrix, respectively, with ξ equal to m and f. We note that the deformations $\varepsilon'^{(\xi\alpha)}_{\alpha\alpha}$ and $\varepsilon'^{(\xi\gamma)}_{\alpha\beta}$ ($\alpha \neq \beta \neq \gamma$) are equal, respectively, to the deformations $\varepsilon^{\xi\alpha}_{\alpha\alpha}$ and $\varepsilon^{\xi\gamma}_{\alpha\beta}$ of the ξ-component, participating in a parallel connection.

The values of the volumetric elastic moduli K_ξ and the shear elastic moduli of the μ_ξ-components are determined, respectively, for the processes (1.4), (1.6) and (1.5), (1.6) in a system of consecutively connected elements

$$3K_\xi \varepsilon^{\xi\beta}_{\alpha\alpha} = \sigma_{\alpha\alpha}, \quad \xi = f, m; \tag{1.11}$$

$$2\mu_\xi \varepsilon^{\xi\alpha}_{\alpha\beta} = \sigma_{\alpha\beta}, \quad \alpha \neq \beta,$$

where $\sigma_{\alpha\alpha}$ and $\sigma_{\alpha\beta}$ are the corresponding parts of the tensor of the stresses, attributed to consecutively connected elements.

The substitution of the first and second equalities of (1.2), (1.10), (1.11), respectively, into the second equalities of (1.8), (1.9) gives

$$\varkappa = \frac{K_f/V_f}{K_f/V_f + K_m/V_m}, \quad \eta = \frac{\mu_f/V_f}{\mu_f/V_f + \mu_m/V_m}. \tag{1.12}$$

Substituting (1.3) into (1.1) and differentiating the result with respect to ε_{ij}^{c}, taking account of relationships (1.8), (1.9), we obtain formulas for the effective Lamé coefficients λ_c, μ_c and the module K_c, as well as the coefficient of linear thermal expansion α_c of a fibrous composite material

$$\begin{gathered} K_c = K_m V_m (1 + 2\varkappa)^2/9 + K_f V_f (3 - 2\varkappa)^2/9; \\ \mu_c = \mu_m V_m (1 + 2\eta^2)/3 + \mu_f V_f [1 + 2(1 - \eta)^2]/3; \\ \lambda_c = K_c - (2/3)\mu_c, \ \alpha_c = \gamma_c/3K_c, \\ \gamma_c = V_m K_m \alpha_m (1 + 2\varkappa) + V_f K_f \alpha_f (3 - 2\varkappa) \end{gathered} \tag{1.13}$$

(the values of $\varkappa$ and η are determined by the relationships (1.12)).

§2. As an example of the application of formula (1.13), let us consider a matrix, i.e., ÉD-6 epoxide resin, hardened with methyltetrahydrophthalic anhydride and reinforced with a polycrystalline boric fiber. In [9], values were obtained for the elastic moduli, depending on T, and the coefficient of linear thermal expansion of the epoxide resin under discussion. We use these data, as well as the elastic moduli of polycrystalline boric fiber [10], neglecting their dependence on T in comparison with the corresponding dependence for the resin. Figures 1-3 show calculated dependences of the elastic moduli of a composite material on the relative volumetric concentration, temperature, and degree of cross-linking r_0/Nb, where r_0 is the distance between the ends of a resin–hardener chain; b is the mean length of a segment; N is the number of segments in a chain. As Fig. 1 shows, the elastic moduli of a composite material are equal to the elastic moduli of a component if the relative concentration of the latter is equal to unity. The course of the curves in Fig. 1 corresponds qualitatively to the fact that a chaotic disorientation of the fibers decreases the elastic modulus of the composite material. The decrease in the elastic moduli (see Figs. 2 and 3) with a rise in the temperature and with a decrease in the degree of cross-linking is also in agreement with the experimental data.

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